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A Measure of the Symmetry of Random Walks

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We derive the probability density for a simple measure of the asymmetry of a one-dimensional random walk, namely the ratio of the minimum of the two maximum displacements in the positive and negative directions, to the maximum. We show that in the diffusion limit the asymmetry is independent of time. These results are shown to apply to random walks in which individual steps have a stable law distribution as well as to multidimensional random walks.

KEY WORDS: Random walks; stable laws; measures of asymmetry.

1. INTRODUCTION

A random walk with symmetric transition probabilities might be thought to exhibit symmetry in all of its significant statistical parameters. That there may be striking deviations from this expected symmetry has been emphasized in the work of Solč and Stockmayer[(1)] and Solč,[(2,3)] although the initial observation is due to Kuhn.[(4)] More recent papers have also examined the same phenomenon, using measures of asymmetry based on the radius of gyrations.[(5–7)] In this paper we analyze a simple measure of anisotropy whose properties can be calculated in closed form when the interval size is large compared to some length characterizing the size of a single step. The anisotropy, as measured by a parameter to be introduced in the next paragraph will be shown to be asymptotically independent of step number, and remains unchanged even when the random walk

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displacements have a stable law distribution in place of a distribution with finite variance increments. Our measure of anisotropy is related to the so-called arcsine law for random walks,[8,9] which has long been a part of the mathematical literature.

Consider first a random walk in one dimension. The maximum displacement in the positive direction after n steps will be denoted by $b(n) \geqslant 0$ and in the negative direction by $a(n)$, where this parameter is also nonnegative. The measure of anisotropy to be analyzed will be denoted by $\rho(n)$, which is the random variable

$$\rho(n) = \min[a(n), b(n)]/\max[a(n), b(n)] \tag{1}$$

so that $\rho(n) \leqslant 1$. When the variance associated with an individual step of the random walk is finite we may pass to the diffusion limit in which the random parameter ρ remains finite with probability one. In the remainder of the paper we calculate the probability density of the parameter ρ, which will be denoted by $g(\rho, t)$, although it will later be shown that for ordinary random walks (i.e., not self-avoiding) $g(\rho, t)$ is independent of time.

2. FINITE VARIANCE WALKS

When the variance σ^2 of an individual step is finite we may pass to the diffusion limit, in which case we start by calculating an expression for the joint density of $a(t)$ and $b(t)$, to be denoted by $p(a, b; t)$. Let $Q(a, b; t)$ denote the probability that a diffusing particle initially at $x = 0$ remains within the interval $(-a, b)$ for a time t. A simple argument then shows that $p(a, b; t)$ can be expressed in terms of $Q(a, b; t)$ as

$$p(a, b; t) = \frac{\partial Q(a, b; t)}{\partial a \, \partial b} \tag{2}$$

Let $\Phi(x)$ denote the error function

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp\left(-\frac{1}{2} u^2\right) du \tag{3}$$

It has been shown[10] that $Q(a, b; t)$ can be expressed as

$$Q(a, b; t) = \sum_{j=-\infty}^{\infty} (-1)^j \left\{ \Phi\left(\frac{a(1+j)+bj}{(2Dt)^{1/2}}\right) + \Phi\left(\frac{aj+b(1+j)}{(2Dt)^{1/2}}\right) \right\} \tag{4}$$

Since this series is formally divergent, we will interpret it as an Abelian sum.[11] That is to say, we will interpret it as

$$Q(a, b; t) = \lim_{x \to 1-} \sum_{j=-\infty}^{\infty} (-1)^j \left\{ \Phi\left(\frac{a(1+j)+bj}{(2Dt)^{1/2}}\right) + \Phi\left(\frac{aj+b(1+j)}{(2Dt)^{1/2}}\right) \right\} x^{|j|} \tag{5}$$

The sum on the right-hand side of this equation is clearly convergent when $|x| < 1$. The combination of Eqs. (2) and (5) and some further algebraic manipulations allow us to express $p(a, b; t)$ in the form

$$p(a, b; t) = \lim_{x \to 1-} \frac{1}{2\pi^{1/2}(Dt)^{3/2}} \times \sum_{j=1}^{\infty} (-1)^{j+1} j(j+1) \left\{ [a(1+j)+bj] \exp\left(-\frac{[a(1+j)+bj]^2}{4Dt}\right) + [b(1+j)+aj] \exp\left(-\frac{[aj+b(1+j)]^2}{4Dt}\right) \right\} x^j \tag{6}$$

Let us first suppose that $a \leqslant b$. The contribution from this possibility to $g(\rho, t)$ can be found by changing variables from (a, b) to (ρ, b), where ρ is defined in Eq. (1), and integrating over all possible values of b. The second contribution comes by a similar calculation from $b \leqslant a$, where we now integrate over all values of a. Thus we write

$$\begin{aligned} g(\rho, t) &= \int_0^\infty ap(a, \rho a; t)\, da + \int_0^\infty bp(\rho b, b; t)\, db \\ &= 2 \lim_{x \to 1-} \sum_{j=1}^{\infty} (-1)^{j+1} j(j+1) \left\{ \frac{1}{[\rho j + j + 1]^2} + \frac{1}{[j + \rho(j+1)]^2} \right\} x^j \end{aligned} \tag{7}$$

It is interesting to verify that $g(\rho)$ defined in Eq. (7) is properly normalized. If we perform term-by-term integration, we find

$$\begin{aligned} \int_0^\infty g(\rho)\, d\rho &= 2 \lim_{x \to 1-} \sum_{j=1}^{\infty} (-1)^{j+1} j(j+1) \left(\frac{1}{j} + \frac{1}{j+1}\right) \frac{x^j}{2j+1} \\ &= 2 \lim_{x \to 1-} \sum_{j=1}^{\infty} (-1)^{j+1} x^j = 2 \lim_{x \to 1-} \frac{x}{x+1} = 1 \end{aligned} \tag{8}$$

We have expressed $g(\rho)$ in terms of a series that is only convergent in the Abelian sense. It is useful to find a numerically computable representation for this function. This can be done adding and subtracting appropriate terms in Eq. (7). Thus, we write

$$g(\rho)=2 \lim_{x \to 1-} \sum_{j=1}^{\infty} (-1)^{j+1} j(j+1) \left\{ \frac{1}{[\rho j+j+1]^2} - \frac{1}{(1+\rho)^2 j^2} \right.$$
$$\left. + \frac{1}{(1+\rho)^2 j^2} + \frac{1}{[j+\rho(j+1)]^2} - \frac{1}{(1+\rho)^2 j^2} + \frac{1}{(1+\rho)^2 j^2} \right\} x^j \tag{9}$$

The second term in the brackets on the right-hand side, for example, has the effect of subtracting out the value of the first term at $j=\infty$, thereby converting the formally divergent sum to a formally convergent one when we set $x=1$. The third and sixth terms in brackets in this last equation can be summed separately from the remaining terms, finally allowing us to express $g(\rho)$ as

$$\begin{aligned} g(\rho) &= \frac{4}{(1+\rho)^2} \lim_{x \to 1-} \sum_{j=1}^{\infty} (-1)^{j+1} \left(1+\frac{1}{j}\right) x^j + \frac{g_1(\rho)}{(1+\rho)^2} \\ &= \frac{2+4 \ln 2}{(1+\rho)^2} + \frac{g_1(\rho)}{(1+\rho)^2} \end{aligned} \tag{10}$$

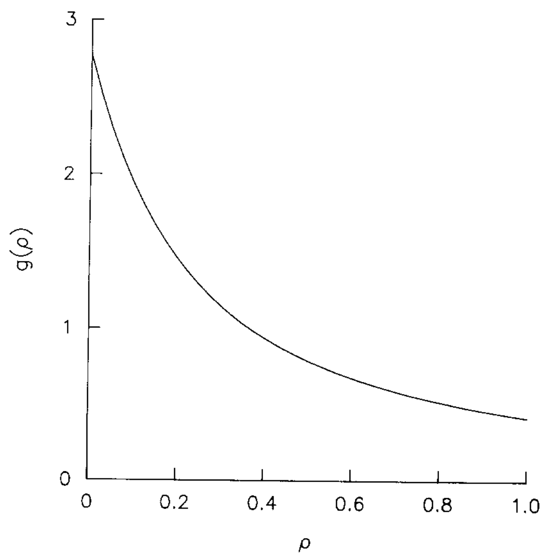

Fig. 1. Plot of $g(\rho)$ as a function of ρ.

in which the function $g_1(\rho)$ can be written as a series convergent in the ordinary sense:

$$g_1(\rho)=2\sum_{j=1}^{\infty}(-1)^j\left(1+\frac{1}{j}\right)\left\{\frac{2j(1+\rho)+1}{[j(1+\rho)+1]^2}+\frac{2j\rho(1+\rho)+\rho^2}{[j(1+\rho)+\rho]^2}\right\} \quad (11)$$

Figure 1 shows a plot of $g(\rho)$ as a function of ρ. The maximum value is at 0 and the function decreases monotonically toward the minimum at $\rho=1$. This behavior is in qualitative agreement with the prediction of the arcsine law. The average value of ρ is found to be approximately 0.3466 with the associated standard deviation $\sigma=0.2761$.

3. STABLE LAW WALKS

A point of further interest is that the form of $g(\rho)$ found in Eq. (6) or Eq. (10) is not peculiar to random walks whose steps have finite variances. Consider a symmetric random walk in which the probability of making a displacement of j sites goes asymptotically like

$$p(j)\sim 1/|j|^{1+\alpha}, \qquad 0<\alpha<1 \quad (12)$$

Working in the limits $a, b\to\infty$, one finds for the function $Q(a, b; n)$

$$Q\sim\frac{1}{\pi}\sum_{j=-\infty}^{\infty}\exp\left(-\frac{\pi^\alpha nk(2j+1)^\alpha}{(a+b)^\alpha}\right)\left\{\frac{\sin[\pi(2j+1)a/(a+b)]}{2j+1}+\frac{\sin[\pi(2j+1)b/(a+b)]}{2j+1}\right\} \quad (13)$$

where k is a constant that plays no further role and without loss of generality may be set equal to 1. In order to obtain results analogous to those in Eqs. (6) and (11), we make a Poisson transformation of this equation, which gives

$$Q(a, b; n)\sim\frac{1}{2}\sum_{j=-\infty}^{\infty}(-1)^j\left\{f\left[\frac{a+j(a+b)}{n^{1/\alpha}}\right]+f\left[\frac{b+j(a+b)}{n^{1/\alpha}}\right]\right\} \quad (14)$$

where

$$f(z)=\int_{-\infty}^{\infty}\exp(-|x|^\alpha)\frac{\sin(zx)}{x}\,dx \quad (15)$$

The reader will notice that the sum in Eq. (14) is again divergent in the ordinary sense if we make use of the estimate $\lim_{z \to \infty} f(z) = \pi$. This divergence is eliminated once again by interpreting Eq. (14) as

$$Q(a, b; n) \sim \frac{1}{2} \lim_{x \to 1-} \sum_{j=-\infty}^{\infty} (-1)^j \left\{ f\left[\frac{a + j(a+b)}{n^{1/\alpha}}\right] + f\left[\frac{b + j(a+b)}{n^{1/\alpha}}\right] \right\} x^{|j|} \tag{16}$$

If we again transform variables from (a, b) to (a, ρ) and (a, b) to (ρ, b) and perform the integrations as in Eq. (6), we obtain results identical to those found for the case of finite-variance random walks, since these depend only on the fact that the argument in Eq. (14) is linear in the variables a and b.

4. THE TWO-DIMENSIONAL DIFFUSION PROCESS

We consider a generalization of the asymmetry parameter ρ in two dimensions, showing that it is independent of time for isotropic diffusion processes or random walks. The same reasoning may be generalized to show that the conclusion is valid in any number of dimensions. Let $Q(\mathbf{a}, \mathbf{b}; t) = Q(a_1, b_1, a_2, b_2; t)$ be the probability that at time t the random walker has remained within a rectangle specified by the four lines $x = a_1$, $x = a_2$, $y = b_1$, $y = b_2$. Then it is easily shown that provided that the diffusion tensor D is diagonal with equal elements, $Q(\mathbf{a}, \mathbf{b}; t)$ is factorable in the sense that

$$Q(\mathbf{a}, \mathbf{b}; t) = Q(a_1, b_1; t)\, Q(a_2, b_2; t) \tag{17}$$

where $Q(\mathbf{a}, \mathbf{b}; t)$ is just the function given in Eq. (5). The joint probability density for $(\mathbf{a}, \mathbf{b})$ at time t is therefore also a product

$$p(\mathbf{a}, \mathbf{b}; t) = \frac{\partial^2 Q(a_1, b_1; t)}{\partial a_1\, \partial b_1} \frac{\partial^2 Q(a_2, b_2; t)}{\partial a_2\, \partial b_2} \tag{18}$$

Without loss of generality we can choose $a_1 = \min(\mathbf{a}, \mathbf{b})$. Three cases must now be considered, depending on whether b_1, b_2, or a_2 is the maximum of all of the displacements. Let us suppose first that b_1 is the maximum. Then the joint density of the minimum displacement a and the maximum displacement b at time t is

$$p_1(a, b; t) = \frac{\partial^2 Q(a, b; t)}{\partial a\, \partial b}\, Q(b - a, b - a; t) \tag{19}$$

which is found from Eq. (17). Similarly, if a_2 is the maximum, we have

$$p_2(a, b; t) = \left.\frac{\partial Q(a, b_1; t)}{\partial a}\right|_{b_1 = b - a} \left.\frac{\partial Q(b, b_2; t)}{\partial b}\right|_{b_2 = b - a} \tag{20}$$

Finally, the third contribution is

$$p_3(a, b; t) = \left.\frac{\partial Q(a, b_1; t)}{\partial a}\right|_{b_1 = b - a} \left.\frac{\partial Q(a_2, b; t)}{\partial b}\right|_{a_2 = b - a} \tag{21}$$

The joint density $p(a, b; t)$ is the sum of the three components given in Eqs. (19)–(21), and the relation between $p(a, b; t)$ and $g(\rho)$ is just that given in the first line of Eq. (6). Since $Q(a, b; t)$ is given by Eq. (4), time-dependent factors multiply the detailed expressions for the derivatives appearing in Eqs. (19)–(21), but these drop out when the integrals over ρ are evaluated, just as in the one-dimensional case. Hence the function $g(\rho; t)$ is again independent of time, although the series that must be evaluated in order to find the detailed form of $g(\rho)$ are much more complicated than in one dimension. We have not investigated the situation when the random walks follow a higher dimensional stable process, but expect that in the appropriate limit these will also exhibit the same "frozen-in" asymmetry. We also expect that asymmetry effects for self-avoiding random walks would be generally much greater than those for the simple random walks of the present paper.

The main point made by the results of this investigation is that there is no unique characterization of the symmetry of a random walk, and the conclusions drawn from the study of one symmetry parameter do not necessarily apply in the case of any other. It would be of some interest to extend the present calculations to other classes of random walks exemplified by SAWs, as has been done by Rudnick and Gaspari.[5]

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